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Some properties of fractional calculus operators for certain analytic functions

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Abstract

Using the fractional calculus operator $D_z^\lambda f(z)$ (fractional derivatives and fractional integrals) for functions $f(z)$ which are analytic in the open unit disk $\mathbb{U}$, a new fractional operator $\Omega^\lambda f(z)$ of $f(z)$ is defined by $\Omega^\lambda f(z) = \Gamma(2-\lambda)z^\lambda D_z^\lambda f(z)$ for any real λ. This operator $\Omega^\lambda f(z)$ is the generalization operator of Sălăgean derivative operator and Libera integral operator for $f(z)$. With this fractional operator $\Omega^\lambda f(z)$, some subclasses of $f(z)$ are defined by subordinations. The object of the present paper is to discuss some problems for functions $f(z)$ belonging to these classes. Finally, a new fractional operator $O_{\gamma.z}^\lambda f(z)$ for $f(z)$ is introduced by using the fractional calculus operator. This new fractional operator is the generalization of some historical operators.

1 Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $f(z) \in \mathcal{A}$, we define the following fractional calculus operator (fractional integrals and fractional derivatives) given by Owa [5] (also by Owa and Srivastava [6]).

Definition 1.1 *The fractional integral of order λ is defined, for a function $f(z) \in \mathcal{A}$, by*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \qquad (\lambda > 0), \tag{1.2}$$

where the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 1.2 *The fractional derivative of order λ is defined, for a function $f(z) \in \mathcal{A}$, by*

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \left\{ \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \right\} \qquad (0 \leqq \lambda < 1), \tag{1.3}$$

where the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

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Definition 1.3 *Under the hypotheses of Definition* 1.2, *the fractional derivative of order* $n+\lambda$ *is defined, for a function* $f(z) \in \mathcal{A}$, *by*

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n}\left(D_z^\lambda f(z)\right) \qquad (0 \leqq \lambda < 1; n = 0, 1, 2, \cdots). \tag{1.4}$$

Remark 1.1 From Definition 1.1, Definition 1.2 and Definition 1.3, we see that

$$D_z^{-\lambda} z^j = \frac{\Gamma(j+1)}{\Gamma(j+\lambda+1)} z^{j+\lambda} \qquad (\lambda > 0),$$

$$D_z^{\lambda} z^j = \frac{\Gamma(j+1)}{\Gamma(j-\lambda+1)} z^{j-\lambda} \qquad (0 \leqq \lambda < 1),$$

and

$$D_z^{n+\lambda} z^j = \frac{\Gamma(j+1)}{\Gamma(j-n-\lambda+1)} z^{j-n-\lambda} \qquad (0 \leqq \lambda < 1; n = 0, 1, 2, \cdots).$$

Therefore, we say that

$$D_z^{\lambda} z^j = \frac{\Gamma(j+1)}{\Gamma(j-\lambda+1)} z^{j-\lambda}$$

for any real λ. This gives us that, for $f(z) \in \mathcal{A}$,

$$D_z^\lambda f(z) = \frac{z^{-\lambda}}{\Gamma(2-\lambda)}\left(z + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)} a_n z^n\right)$$

for any real λ.

In view of Remark 1.1, we introduce the following fractional operator $\Omega^\lambda f(z)$ for $f(z) \in \mathcal{A}$ by

$$\begin{aligned} \Omega^\lambda f(z) &= \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)} a_n z^n \end{aligned} \tag{1.5}$$

for any real λ and

$$\begin{aligned} \Omega^{\lambda_1+\lambda_2} f(z) &= \Gamma(2-\lambda_1-\lambda_2) z^{\lambda_1+\lambda_2} D_z^{\lambda_2}\left(D_z^{\lambda_1} f(z)\right) \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda_1-\lambda_2)\Gamma(n+1)}{\Gamma(n-\lambda_1-\lambda_2+1)} a_n z^n \\ &= \Omega^{\lambda_2+\lambda_1} f(z) \end{aligned} \tag{1.6}$$

for any real λ_1 and λ_2.

Remark 1.2 We note that

$$\Omega^0 f(z) = f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

$$\Omega^1 f(z) = \Omega f(z) = z f'(z) = z + \sum_{n=2}^{\infty} n a_n z^n,$$

and

$$\Omega^j f(z) = \Omega\left(\Omega^{j-1} f(z)\right) = z + \sum_{n=2}^{\infty} n^j a_n z^n \qquad (j = 1, 2, 3, \cdots)$$

which was called Sălăgean derivative operator introduced by Sălăgean [7]. Also we see that

$$\Omega^{-1} f(z) = \frac{2}{z}\int_0^z f(t)dt = z + \sum_{n=2}^{\infty} \frac{2}{n+1} a_n z^n$$

and

$$\Omega^{-j} f(z) = \Omega^{-1}\left(\Omega^{-j+1} f(z)\right) = z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^j a_n z^n \qquad (j = 1, 2, 3, \cdots)$$

which was called Libera integral operator defined by Libera [4]. Thus, our operator $\Omega^{\lambda} f(z)$ is the generalization operator of Sălăgean derivative operator and Libera integral operator.
Libera integral operator is generalized as Bernardi integral operator given by Bernardi [1] as follows:

$$\frac{1+\gamma}{z^{\gamma}}\int_0^z f(t)t^{\gamma-1}dt = z + \sum_{n=2}^{\infty} \frac{1+\gamma}{n+\gamma} a_n z^n \qquad (\gamma = 1, 2, 3, \cdots).$$

This means that our fractional operator and Bernardi integral operator are the generalization of Libera integral operator.

2 Properties of the class $\mathcal{A}(\alpha, \beta, \gamma; \lambda)$

For two analytic functions $f(z)$ and $g(z)$ in $\mathbb{U}$, $f(z)$ is said to be subordinate to $g(z)$, written $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ which satisfies $w(0) = 0, |w(z)| < 1$ $(z \in \mathbb{U})$, and $f(z) = g(w(z))$. If $g(z)$ is univalent in $\mathbb{U}$, then this subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ (cf. see Duren [3]).

Let us define the subclass $\mathcal{A}(\alpha, \beta, \gamma; \lambda)$ of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

$$\alpha\frac{\Omega^{\lambda} f(z)}{z} + \beta\frac{\Omega^{1+\lambda} f(z)}{z} \prec \frac{1+(1-2\gamma)z}{1-z} \qquad (z \in \mathbb{U}) \tag{2.1}$$

for some real $\alpha(\alpha > 0), \beta(\beta > 0)$, and $\gamma(0 \leqq \gamma < \alpha + \beta)$.

For $f(z) \in \mathcal{A}(\alpha, \beta, \gamma; \lambda)$, we have

Theorem 2.1 *A function $f(z) \in \mathcal{A}$ is in the class $f(z) \in \mathcal{A}(\alpha, \beta, \gamma; \lambda)$ if and only if*

$$f(z) = z + \frac{2(\alpha+\beta-\gamma)}{\Gamma(2-\lambda)}\int_{|x|=1}\left(\sum_{n=2}^{\infty} \frac{\Gamma(n+1-\lambda)}{n!(\alpha+n\beta)} z^n\right) d\mu(x), \tag{2.2}$$

where $\mu(x)$ is the probability measure on $X = \{x \in \mathbb{C} : |x| = 1\}$.

Corollary 2.1 *If $f(z) \in \mathcal{A}(\alpha, \beta, \gamma; \lambda)$, then*

$$|a_n| \leqq \frac{2(\alpha+\beta-\gamma)|\Gamma(n+1-\lambda)|}{n!(\alpha+n\beta)|\Gamma(2-\lambda)|} \qquad (n \geqq 2). \tag{2.3}$$

Equality holds true for $f(z)$ given by

2.4) $$f(z) = z + \frac{2(\alpha+\beta-\lambda)}{\Gamma(2-\lambda)}\left(\sum_{n=2}^{\infty}\frac{\Gamma(n+1-\lambda)}{n!(\alpha+n\beta)}z^n\right).$$

Next, we derive

Theorem 2.2 *If $f(z) \in \mathcal{A}(\alpha,\beta,\gamma;\lambda)$, then*

(2.5) $$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1-\mu$$

for $|z| < r_0$, where

(2.5) $$r_0 = \inf_{n\geqq 2}\left(\frac{(n-2)!(1-\mu)(\alpha+n\beta)|\Gamma(2-\lambda)|}{2(n-\mu)(\alpha+\beta-\gamma)|\Gamma(n+1-\lambda)|}\right)^{\frac{1}{n-1}} \qquad (0 \leqq \mu < 1).$$

Therefore, $f(z)$ is starlike of order μ for $|z| < r_0$.

Theorem 2.3 *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty}\left(\sum_{j=1}^{m}\frac{\alpha_j|\Gamma(2-\lambda_j)|}{|\Gamma(n+1-\lambda_j)|}\right)n!|a_n| \leqq \sum_{j=1}^{m}\alpha_j - \beta$$

for some real $\alpha_j(\alpha_j \geqq 0), \lambda_j$, and $\beta(0 \leqq \beta < \sum_{j=1}^{m}\alpha_j)$, then

$$\mathrm{Re}\left(\sum_{j=1}^{m}\alpha_j\frac{\Omega^{\lambda_j}f(z)}{z}\right) \prec \frac{1+(1-2\beta)z}{1-z} \qquad (z \in \mathbb{U}).$$

3 Properties for the classes $\mathcal{S}^*_\lambda$ and $\mathcal{K}_\lambda$

Let us consider the following linear transformation w of ζ for a fixed $z \in \mathbb{U}$ by

(3.1) $$w = w(\zeta) = \frac{z+\zeta}{1+\bar{z}\zeta} \qquad (z \in \mathbb{U}).$$

Then, we observe that $|\zeta| < 1$ corresponds to $|w| < 1$ and $\zeta = 0$ corresponds to $w = z$. Letting $F(z) = \Omega^\lambda f(z)$, we introduce

(3.2) $$g(\lambda;\zeta) = \frac{F(w)-F(z)}{F'(z)(1-|z|^2)} \qquad (\zeta \in \mathbb{U}),$$

where w is given by (3.1). It follows that $g(\lambda;0) = 0$ and $g'(\lambda;0) = 1$. This implies that $g(\lambda;\zeta) \in \mathcal{A}$ if $f(z) \in \mathcal{A}$. For $f(z) \in \mathcal{A}$, we say that $f(z) \in \mathcal{S}^*_\lambda$ if $f(z)$ satisfies

(3.3) $$\frac{\Omega^{1+\lambda}f(z)}{\Omega^\lambda f(z)} \prec \frac{1+z}{1-z} \qquad (z \in \mathbb{U}).$$

Further, let $f(z) \in \mathcal{K}_\lambda$ if $f(z)$ satisfies $\Omega^{1+\lambda}f(z) \in \mathcal{S}^*_\lambda$.

Now, we derive

Theorem 3.1 *If $f(z) \in \mathcal{S}^*_\lambda$, then*

$$\left|D_z^n \Omega^\lambda f(z)\right| \leqq \frac{n!(n+|z|)}{(1-|z|)^{n+2}} \qquad (z \in \mathbb{U}) \tag{3.4}$$

for $n = 0, 1, 2, \cdots$. Equality holds true for $f(z)$ defined by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n)} z^n.$$

Corollary 3.1 *If $f(z) \in \mathcal{S}^*_\lambda$, then*

$$|D_z^\lambda f(z)| \leqq \frac{|z|}{|z|^\lambda(1-|z|)^2|\Gamma(2-\lambda)|},$$

$$|D_z^{1+\lambda} f(z)| \leqq \frac{1}{|z|^\lambda(1-|z|)^2|\Gamma(2-\lambda)|}\left(|\lambda| + \frac{1+|z|}{1-|z|}\right),$$

and

$$|D_z^{2+\lambda} f(z)| \leqq \frac{1}{|z|^\lambda(1-|z|)^2|\Gamma(2-\lambda)|}\left(\frac{|\lambda(\lambda-1)|}{|z|} + \frac{2|\lambda|}{|z|}\left(|\lambda| + \frac{1+|z|}{1-|z|}\right) + \frac{2(2+|z|)}{(1-|z|)^2}\right)$$

for $z \in \mathbb{U}$.

Corollary 3.2 *If $f(z) \in \mathcal{S}^*_0$, then*

$$|f^{(n)}(z)| \leqq \frac{n!(n+|z|)}{(1-|z|)^{n+2}} \qquad (z \in \mathbb{U}). \tag{3.5}$$

Equality is attended for Keobe function $f(z) = \dfrac{z}{(1-z)^2}$.

Theorem 3.2 *If $f(z) \in \mathcal{K}_\lambda$, then*

$$\left|D_z^n \Omega^\lambda f(z)\right| \leqq \frac{n!}{(1-|z|)^{n+1}} \qquad (z \in \mathbb{U}) \tag{3.6}$$

for $n = 0, 1, 2, \cdots$. Equality is attended for $f(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)} z^n.$$

Corollary 3.3 *If $f(z) \in \mathcal{K}_\lambda$, then*

$$|D_z^\lambda f(z)| \leqq \frac{|z|}{|z|^\lambda(1-|z|)|\Gamma(2-\lambda)|},$$

$$|D_z^{1+\lambda}f(z)| \leqq \frac{1}{|z|^\lambda(1-|z|)|\Gamma(2-\lambda)|}\left(|\lambda|+\frac{1}{1-|z|}\right),$$

and

$$|D_z^{2+\lambda}f(z)| \leqq \frac{1}{|z|^\lambda(1-|z|)|\Gamma(2-\lambda)|}\left(\frac{|\lambda(\lambda-1)|}{|z|}+\frac{2|\lambda|}{|z|}\left(|\lambda|+\frac{1}{1-|z|}\right)+\frac{2}{(1-|z|)^3}\right)$$

for $z \in \mathbb{U}$.

Corollary 3.4 *If* $f(z) \in \mathcal{K}_0$, *then*

$$|f^{(n)}(z)| \leqq \frac{n!}{(1-|z|)^{n+1}} \qquad (z \in \mathbb{U}).$$

Equality is attended for the function $f(z) = \dfrac{z}{(1-z)}$.

4 A new factional operator concerning with some integral operators

Let us define a new fractional operator $O_{\gamma,z}^{\lambda}f(z)$ by

$$O_{\gamma,z}^{\lambda}f(z) = \frac{\Gamma(\gamma+1-\lambda)}{\Gamma(\gamma+1)}z^{1+\lambda-\gamma}D_z^{\lambda}\left(z^{\gamma-1}f(z)\right) \tag{4.1}$$

$$= z + \sum_{n=2}^{\infty}\frac{\Gamma(\gamma+1-\lambda)\Gamma(n+1)}{\Gamma(\gamma+1)\Gamma(n+\gamma-\lambda)}a_n z^n$$

for any real λ and γ.

$$O_{\gamma,z}^{\lambda_1+\lambda_2}f(z) = \frac{\Gamma(\lambda+1-\lambda_1-\lambda_2)}{\Gamma(\gamma+1)}z^{1+\lambda_1+\lambda_2-\gamma}D_z^{\lambda_2}\left(D_z^{\lambda_1}\left(z^{\gamma-1}f(z)\right)\right) \tag{4.2}$$

$$= z + \sum_{n=2}^{\infty}\frac{\Gamma(\gamma+1-\lambda_1-\lambda_2)\Gamma(n+\gamma)}{\Gamma(\gamma+1)\Gamma(n+\gamma-\lambda_1-\lambda_2)}a_n z^n$$

$$= O_{\gamma,z}^{\lambda_2+\lambda_1}f(z)$$

for any real λ_1, λ_2 and γ.

Remark 4.1 From the definition for the fractional operator $O_{\gamma,z}^{\lambda}f(z)$, we see that

(1) If $\gamma = 1$ and $\lambda = 1$, then we have Sălăgean differential operator [7] :

$$O_{1,z}^{1}f(z) = zf'(z) = z + \sum_{n=2}^{\infty} n a_n z^n$$

(2) If $\gamma = 0$ and $\lambda = -1$, then we have Alexander integral operator [1] :

$$O_{0,z}^{-1}f(z) = \int_0^z \frac{f(t)}{t}dt = z + \sum_{n=2}^{\infty}\frac{1}{n}a_n z^n$$

(3) If $\gamma = 1$ and $\lambda = -1$, then we have Libera integral operator [4] :

$$O_{1,z}^{-1} f(z) = \frac{2}{z} \int_0^z f(t)dt = z + \sum_{n=2}^{\infty} \frac{2}{n+1} a_n z^n$$

(4) If $\lambda = -1$, then we have Bernardi integral operator [2] :

$$O_{\gamma,z}^{-1} f(z) = \frac{1+\gamma}{z^{\gamma}} \int_0^z t^{\gamma-1} f(t)dt = z + \sum_{n=2}^{\infty} \frac{1+\gamma}{n+\gamma} a_n z^n.$$

In view of Remark 4.1, we know that our fractional operator $O_{\gamma,z}^{\lambda} f(z)$ is the generalization of some historical operators (differential operators and integral operators). Therefore, by studying this fractional operator, we get many results connecting with some operators.

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